Symmetry breaking in one-dimensional diffusion

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The mean free passage time for one-dimensional diffusion on a line segment under the influence of a deterministic telegraph signal proves to be a nonmonotonic function of the signal rate (''stochastic resonance'') if symmetry breaking takes place. The symmetry breaking may be expressed either in nonsymmetric boundary conditions (one end absorbing and the other reflecting) or in a nonsymmetric telegraph signal. The latter case is considered in detail. It turns out that the larger the asymmetry of a telegraph signal, the wider the range of parameters for which a stochastic resonance occurs.

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Stochastic resonance (SR) and related phenomena of resonance activation, coherent stochastic resonance, etc. have recently received much attention [1]. All these phenomena share the common property that an output signal or some function of it exhibits nonmonotonic behavior as a function of some characteristics of noise or of a periodic signal. Although as a rule, SR occurs in nonlinear systems driven by a random and a periodic force, it manifests itself in some linear systems as well. A simple example of SR in linear systems is that of one-dimensional diffusion on a segment terminated by one or two traps, where the mean free passage time (MFPT) to be trapped by the boundary (or boundaries) varies nonmonotonically as a function of both the frequency [2] and amplitude [3] of the periodic force. The characteristic frequency needed for ''resonance'' is supplied either by an additional sinusoidal or rectangular pulse signal or by the rate of additional (say, dichotomous) colored noise.

In spite of the serious effort made in the study of resonant phenomena in systems subject to the random force, the globally necessary conditions for the occurrence of SR have not been identified. It becomes clear, however, that some sort of ''symmetry breaking'' is required in the dynamic system in order for it to exhibit SR. We use a linear system to illustrate the importance of symmetry breaking, starting from symmetric and nonsymmetric boundary conditions. If a diffusive particle is exposed to dichotomous noise, the dependence of the MFPT on the noise rate remains monotonic if both boundaries are absorbing [4], and exhibits nonmonotonic, resonance behavior when the ''symmetry'' of the boundary conditions is broken, namely, one of the boundaries is absorbing and the other reflecting [5]. The same effect exists in the well-known problem of jumps in a linear double-well potential when the slope of the potential fluctuates randomly between two values at a rate γ. (These two problems are, in fact, completely isomorphic being described by the same differential equations.) It turns out [6] that the nonmonotonic dependence of the MFPT on γ exists only when the boundary conditions are different at the two end points. The third example, which demonstrates the importance of the symmetry of the boundary conditions, is related to the same problem of a one-dimensional diffusion. However, in this case the random dichotomous noise is replaced by the deterministic rectangular telegraph signal. Again, the MFPT remains a monotonic function of the rate of the telegraph signal if the two boundaries are absorbing [7], and becomes nonmonotonic when one of the boundaries is absorbing and the other is reflecting [8].

Changes in the boundary conditions are not the only path to symmetry breaking. The conjecture was made [8] that the nonmonotonic dependence of the MFPT on the rate of the deterministic telegraph signal may appear even for two absorbing boundary conditions, provided that the symmetry breaking occurs via the replacing of the symmetric rectangular signal by a nonsymmetric signal. It is the aim of this paper to verify this conjecture.

The evolution of the one-dimensional system considered in the overdamped regime is governed by the following Langevin equation:

$$\frac{dx}{dt}=\xi(t)+f(t), \tag{1}$$

where $\xi(t)$ is a zero-mean Gaussian white noise of strength D, i.e., $\langle\xi(t)\xi(t_1)\rangle=2D\,\delta(t-t_1)$, and $f(t)$ is the nonsymmetric telegraph signal,

$$f(t)=\begin{cases} v_1 & \text{for } t\varepsilon[2n\Gamma,(2n+1)\Gamma] \\ -v_2 & \text{for } t\varepsilon[(2n+1)\Gamma,(2n+2)\Gamma], \end{cases} \tag{2}$$

where Γ is the period of the telegraph signal and $n=0,1,2,\ldots$. The rate (frequency) of the signal is given by $\omega=(2\Gamma)^{-1}$.

The Fokker-Planck equation that corresponds to the Langevin equation (1),

$$\frac{\partial p_{1,2}(x,t)}{\partial t}=-f(t)\frac{\partial p_{1,2}(x,t)}{\partial x}+D\frac{\partial^2 p_{1,2}(x,t)}{\partial x^2} \tag{3}$$

subject to the initial condition $p_1(x,t=0|x_0)=\delta(x-x_0)$ and to the absorbing boundary conditions at both ends of the interval $[0,L]$, $p_{1,2}(x=0,t)=p_{1,2}(x=L,t)=0$, has the well-known solution of the form [9]

$$p_1(x,t)=\frac{2}{L}\exp\left[\frac{v_1(x-x_0)}{2D}-\frac{v_1^2 t}{4D}\right]\times\sum_{n=1}^{\infty}\sin(\beta_n x)\sin(\beta_n x_0)\exp(-D\beta_n^2 t), \tag{4}$$

$$p_2(x,t)=\frac{2}{L}\exp\left[-\frac{v_2(x-x_0)}{2D}-\frac{v_2^2 t}{4D}\right]\times\sum_{n=1}^{\infty}\sin(\beta_n x)\sin(\beta_n x_0)\exp(-D\beta_n^2 t),$$

where $\beta_n=\pi n/L$.

The probability density function $p(x,t)$ describes all properties of the random process $x(t)$ including the MFPT, T. However, the standard formula $T=\int_0^\infty dt\int_0^L dx p(x,t)$ [9] becomes slightly more complicated because the general solution of our problem can be obtained by matching the solutions (4) at times $t=k\Gamma$, $k=0,1,2,\ldots$. The probability $P_k(x)$ that at $t=k\Gamma$, the system is found at positions $x,x+dx$ is defined by the obvious recurrence relations

$$P_{m+1}(x)=\int_0^L dy\, p_{1,2}(x,\Gamma|y)P_m(y), \tag{5}$$

where indices 1 and 2 are related to $m=2n$ and $m=2n+1$, respectively. The initial condition $P_0(y)=\delta(y-y_0)$ leads to

$$P_1(y)=p_1(y,\Gamma|x_0). \tag{6}$$

Then, one immediately obtains [10,7],

$$T=\sum_{n=1}^{\infty}\int_0^\Gamma dt\int_0^L dx\int_0^L dy[p_1(x,t|y)P_{2n}(y)+p_2(x,t|y)P_{2n+1}(y)]. \tag{7}$$

Although Eqs. (4)–(7) define, in principle, the full solution of our problem, some approximate procedure has to be used. In order to find whether T is a monotonic function of ω, we apply the following method [8]. First, we shall find the exact limiting values of T for zero and infinite ω. Then, we calculate some approximate asymptotic values of T for small ω. If T approaches the largest (smallest) limit value from above (below), the nonmonotonic dependence $T(\omega)$ will be the geometric necessity.

For $\omega\to\infty$, the fast oscillating signal does not influence the MFPT, which is defined only by pure diffusion, and which turns out to be equal for two absorbing boundaries [9]

$$T(\omega\to\infty)=\frac{x_0(L-x_0)}{2D}. \tag{8}$$

For the opposite limit, $\omega=0$, the MFPT for a system driven by white noise and constant bias v_1 is also well known [9],

$$T(\omega=0)=-\frac{x_0}{v_1}+\frac{L\left[1-\exp\left(-\frac{v_1x_0}{D}\right)\right]}{v_1\left[1-\exp\left(-\frac{v_1L}{D}\right)\right]}. \tag{9}$$

The expression for $T(\omega=0)$ can also be obtained from the general expression (7) with $\Gamma=\infty$ by substituting Eqs. (4) and (5) into Eq. (7). This gives

$$\begin{aligned}T(\omega=0)&=\sum_{n=1}^{\infty}\int_0^\infty dt\int_0^L dx\, p_1(x,t|x_0)\\&=\frac{2D}{L}\exp\left(-\frac{v_1x_0}{2D}\right)\sum_{n=1}^{\infty}\sin(\beta_n x_0)\\&\quad\times\frac{\beta_n\left[1-(-1)^n\exp\left(\frac{v_1L}{2D}\right)\right]}{\left(D\beta_n^2+\frac{v_1^2}{4D}\right)^2}.\end{aligned} \tag{10}$$

When the latter result is compared with that of Eq. (8), it is apparent that $T(\omega=0)<T(\omega\to\infty)$.

Turning now to the calculation of the small correction in ω to Eq. (9), it must be emphasized that, after integration over t, the rate $\omega\equiv(2\Gamma)^{-1}$ appears as strong exponential dependence, $\exp[-(D\beta_n^2+v_1^2/4D)/2\omega]$. The latter allows us to retain only the $n=1$ term in Eq. (7) which, after using Eq. (6), can be rewritten as

$$T(\omega\to 0)\approx\int_0^\Gamma dt\int_0^L dx\, p_1(x,t|x_0)+\int_0^\Gamma dt\int_0^L dx\int_0^L dy\, p_2(x,t|y)p_1(y,\Gamma|x_0). \tag{11}$$

For the same reason, it justified to retain only the $n=1$ term in the p_1 function in Eq. (11), although the sum entering the p_2 function has to be retained. On substituting Eq. (4) into Eq. (11), one can rewrite the leading term in the latter equation in the form

$$T(\omega\to 0)=\frac{2}{L}\exp\left(-\frac{v_1x_0}{2D}\right)\exp\left(-\frac{v_1^2L^2+4\pi^2D^2}{8D\omega L^2}\right)\sin\left(\frac{\pi x_0}{L}\right)\left[-\frac{\pi\left[1+\exp\left(-\frac{v_1L}{2L}\right)\right]}{LD\left(\frac{v_1^2}{4D^2}+\frac{\pi^2}{L^2}\right)^2}+\int_0^L dy\exp\left(\frac{v_1y}{2D}\right)\sin\left(\frac{\pi y}{L}\right)\sum_{n=1}^{\infty}\sin(\beta_n y)\frac{\beta_n\left[1-(-1)^n\exp\left(-\frac{v_2L}{2D}\right)\right]}{\left(D\beta_n^2+\frac{v_2^2}{4D}\right)^2}\frac{2D}{L}\exp\left(\frac{v_2y}{2D}\right)\right]. \tag{12}$$

There is no need to calculate the sum over n in Eq. (12) since the same sum—with v_1 replaced by $-v_2$—appears in Eq.

(10), and one can replace this sum by the equivalent expression from Eq. (9). Equation (12) then contains only trivial integrations. Performing the simple algebra, yields

$$T(\omega\to 0)=\frac{32\pi L^2D^2\left(1+\frac{v_1}{v_2}\right)}{v_1^2L^2+4\pi^2D^2}\exp\left[-\frac{v_1x_0}{2D}-\frac{v_1^2L^2+4\pi^2D^2}{8DL^2\omega}\right]\sin\left(\frac{\pi x_0}{L}\right)$$
$$\times\left\{\frac{v_2L\left[1+\exp\left[\frac{(v_1+2v_2)L}{2D}\right]\right]}{\left[\exp\left(\frac{v_2L}{D}\right)-1\right]\left[v_1^2L^2\left(1+\frac{2v_2}{v_1}\right)^2+4\pi^2D^2\right]}-\frac{D\left[1+\exp\left(\frac{v_1L}{2D}\right)\right]}{v_1^2L^2+4\pi^2D^2}\right\}. \tag{13}$$

From the outset one could use dimensionless units of length and time, which is equivalent to setting $D=v_1=1$ in the final Eq. (13). Introducing the parameter of nonsymmetry of the telegraph signal, $z=v_2/v_1$, one can rewrite Eq. (13) as

$$T(\omega\to 0)=\frac{32\pi L^2(1+z)}{(L^2+4\pi^2)z}\exp\left[-\frac{x_0}{2}-\frac{L^2+4\pi^2}{8L^2\omega}\right]\sin\left(\frac{\pi x_0}{L}\right)\left\{\frac{Lz\left[1+\exp\left(\frac{(1+2z)L}{2}\right)\right]}{[\exp(Lz)-1][L^2(1+2z)^2+4\pi^2]}-\frac{\left[1+\exp\left(\frac{L}{2}\right)\right]}{L^2+4\pi^2}\right\}. \tag{14}$$

Since $T(\omega=0)<T(\omega\to\infty)$, nonmonotonic dependence $T(\omega)$ will certainly occur when $T(\omega\to 0)$ is negative, where the sign of the bracket in Eq. (14) completely determines the sign of $T(\omega\to 0)$. For $z=1$ (symmetric telegraph signal) both this bracket and $T(\omega\to 0)$ are positive, and there is no geometric necessity for nonmonotonic changes of $T(\omega)$. It is just this case that was considered previously [7]. However, a simple analysis shows that the expression (14) becomes negative for $z\geqslant 1.5$, i.e., when the telegraph signal becomes sufficiently nonsymmetric. Moreover, the negative value of Eq. (14) and, hence, the nonmonotonic behavior of $T(\omega)$, occur only for not too large length L of the segment, $L<L_b$, where L_b is some characteristic length which increases with z so that for $z=1.5$, $L_b=5$, and for $z=2.0$, $L_b=8$, etc.

In conclusion, we found that the MFPT of a particle diffusing on a segment and subject to the deterministic telegraph signal, remains the monotonic function of the rate of a signal if the system is ''symmetric,'' i.e., both boundary conditions are absorbing and the telegraph signal has a symmetric rectangular form [7]. However, the violation of either of these conditions (symmetry breaking) results in the appearance of nonmonotonic behavior (SR). It was proved in the previous work [8] that SR occurs for a symmetric signal with nonsymmetric end points (one absorbing and one reflecting). In this paper, we have shown that SR appears for symmetric boundary conditions but with a nonsymmetric telegraph signal.

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